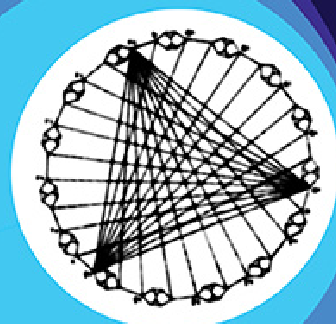
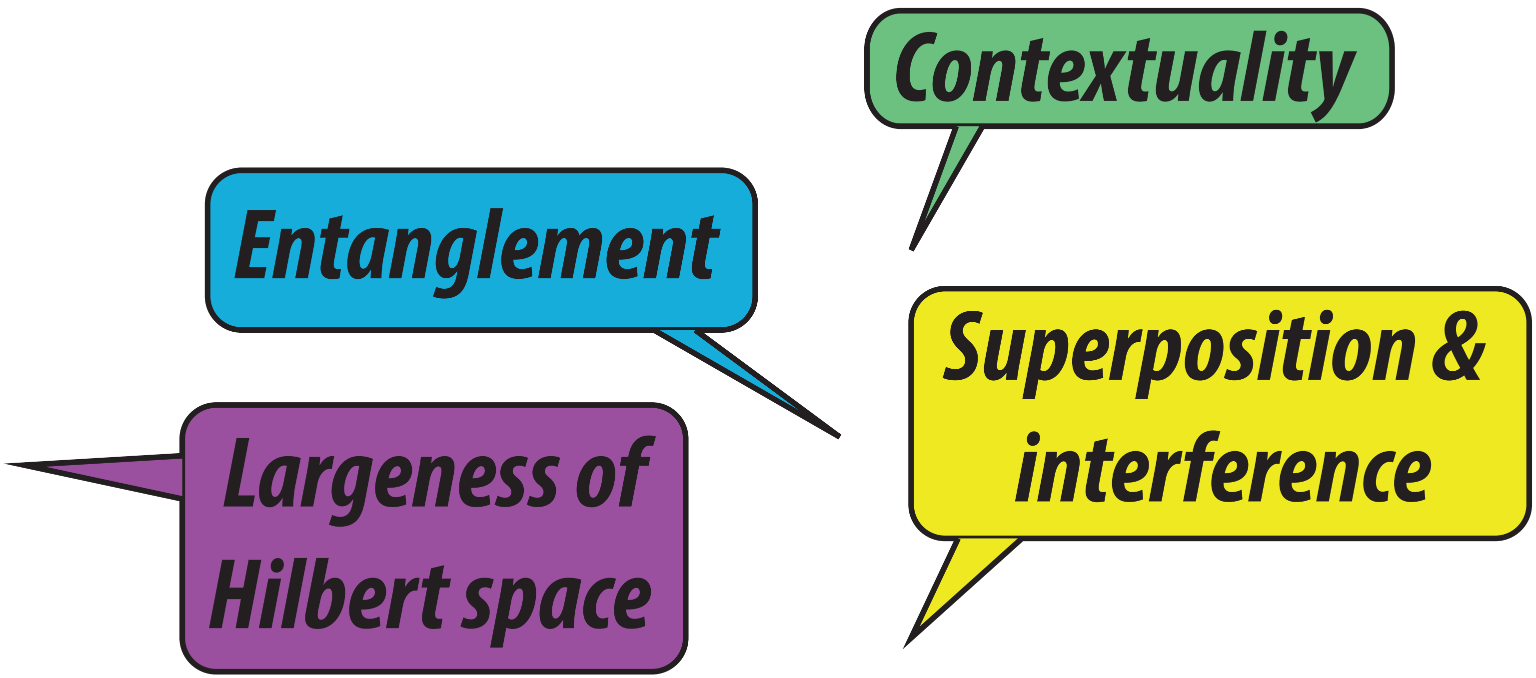


Contextuality and Wigner negativity in Quantum Computation on rebits

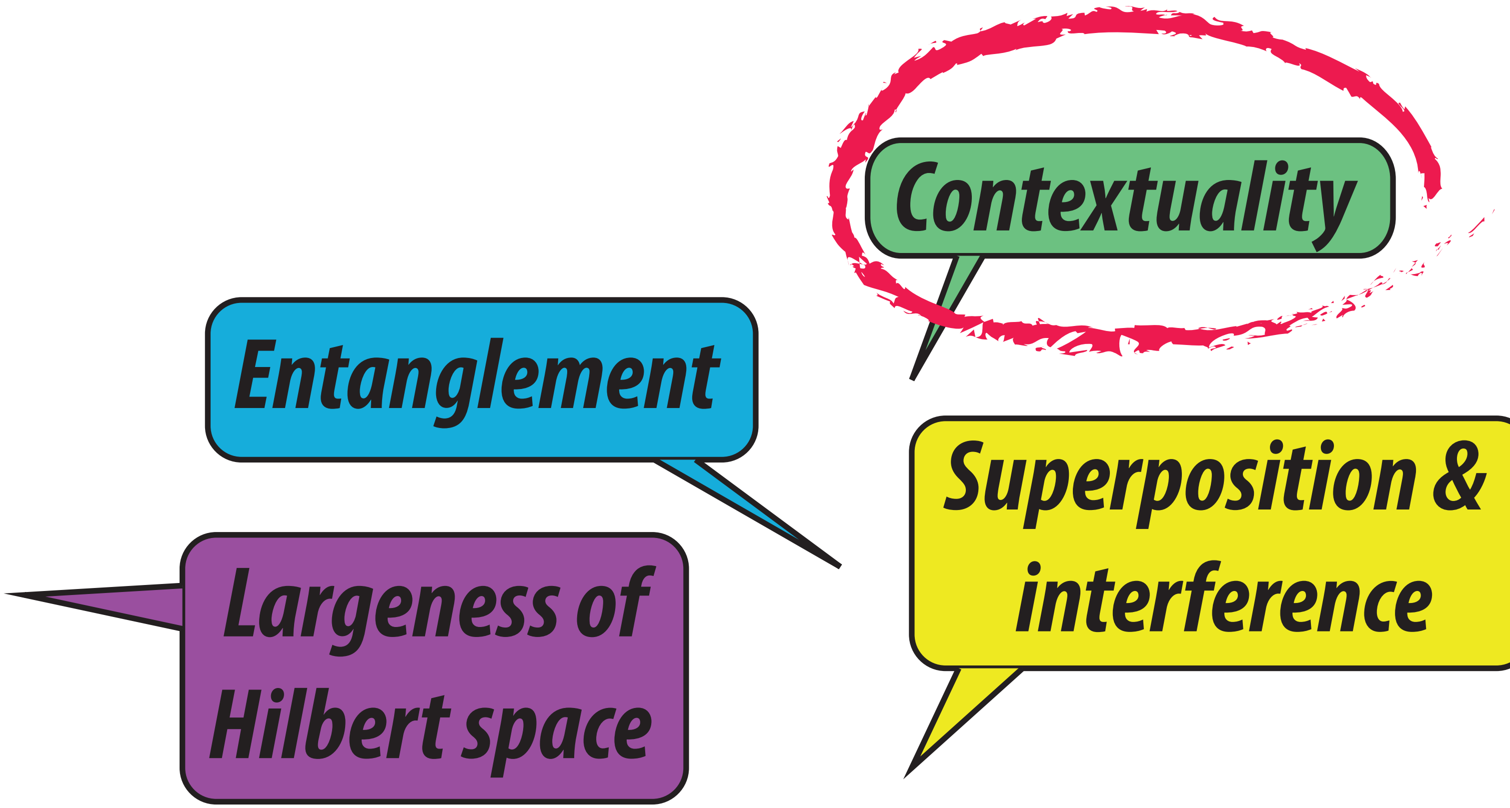
Nicolas Delfosse, Philippe Allard, Jake Bian and Robert Raussendorf

QIP Sydney, January 2015

What makes quantum computing work?

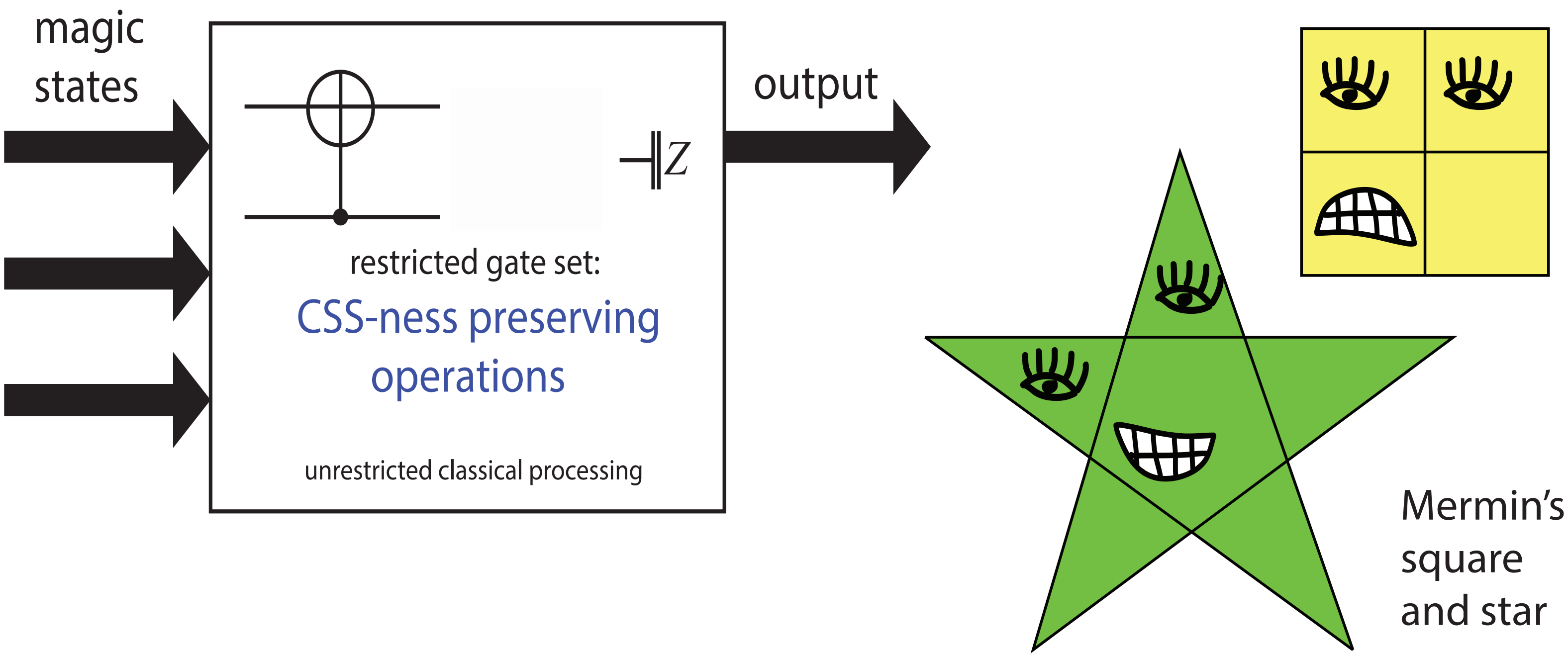


What makes quantum computing work?



What makes quantum computing work?

Result



Contextuality is a necessary resource for universal quantum computation with magic states on rebits

Contextuality in quantum computation

- 1996. DiVincenzo & Peres: *Quantum codewords contradict local realism*

- 2009. Anders & Browne: *Contextuality powers measurement-based quantum computation*

- 2014. Howard *et al.*: *Contextuality powers quantum computation with magic states*

- This talk: *Contextuality provides state magic for rebits*

Outline

1. Review

   (a) Hidden variable models & contextuality

   (b) Quantum computation with magic states

   (c) Wigner functions

2. Quantum computation with magic states on rebits

   (a) The trouble with qubits

   (b) Computational scheme and matching Wigner function

   (c) Negativity and contextuality as resources

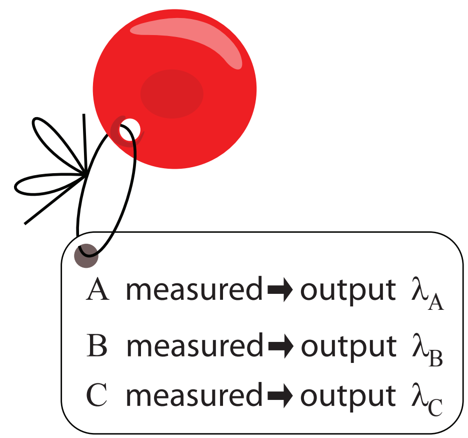
Contextuality of QM

What is a non-contextual hidden-variable model?

quantum mechanics

$$|\Psi\rangle$$

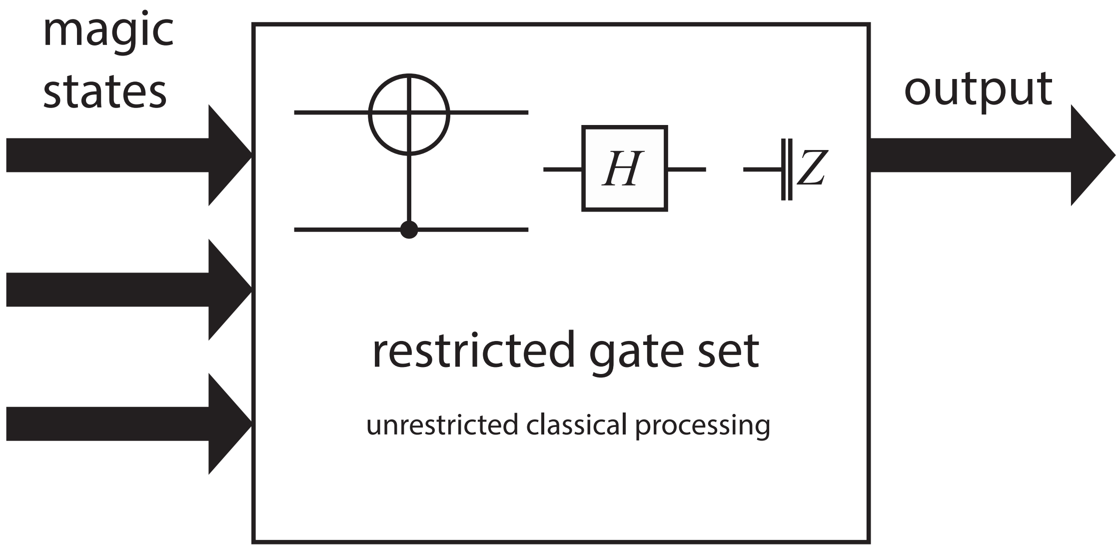
hidden-variable model



Noncontextuality: Given observables A,B,C: $[A,B]=[A,C]=0$:
λ_A is *independent* of whether A is measured jointly with B or C.

Theorem [Kochen, Specker]: For $\dim(\mathcal{H}) \geq 3$, quantum-mechanics cannot be reproduced by a non-contextual hidden-variable model.

Quantum Computation by state injection

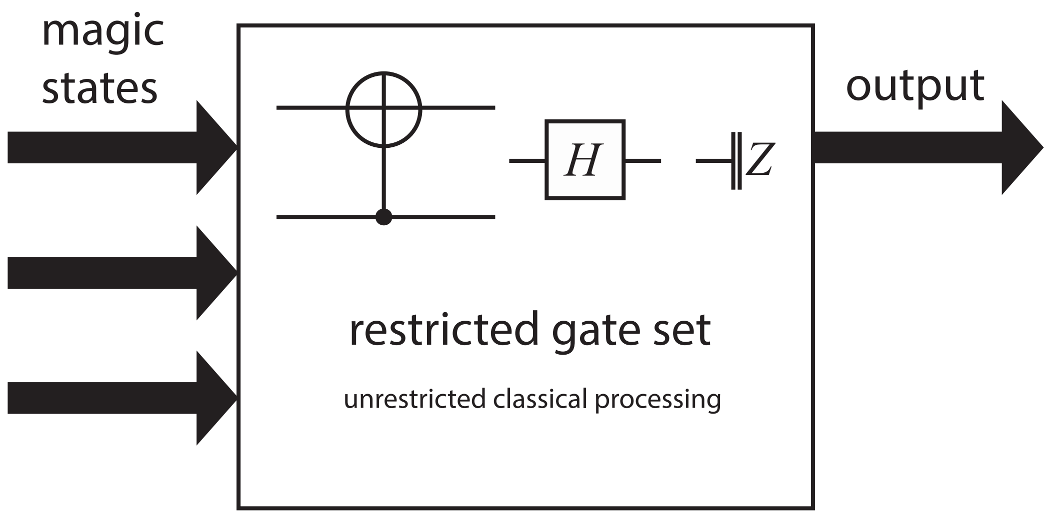


- Non-universal restricted gate set: *e.g. Clifford gates*.
- Universality reached through injection of *magic states*.

\+ *As of now, leading scheme for fault-tolerant QC.*

Computational power is pushed from gates to states

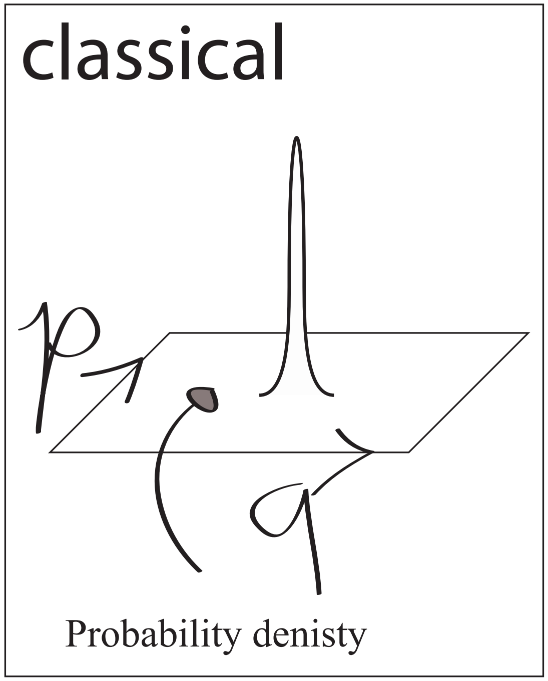
Quantum computation by state injection



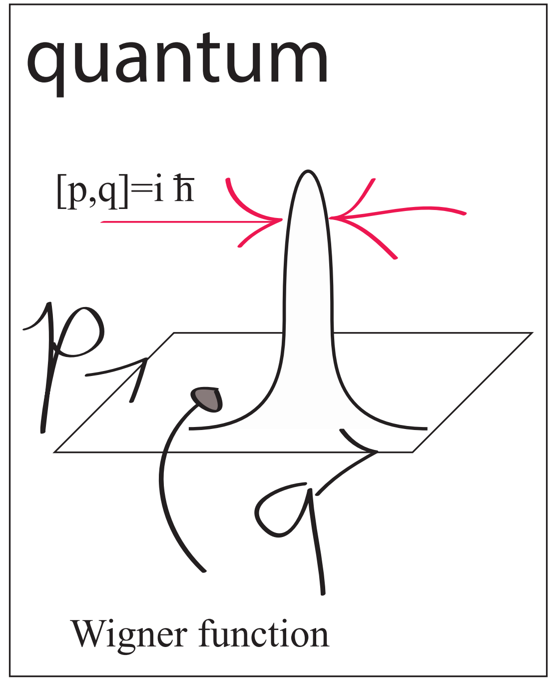
Which properties must the magic states have to enable universality?

A: Wigner function negativity, contextuality

[quantum] mechanics in phase space



Probability denisty



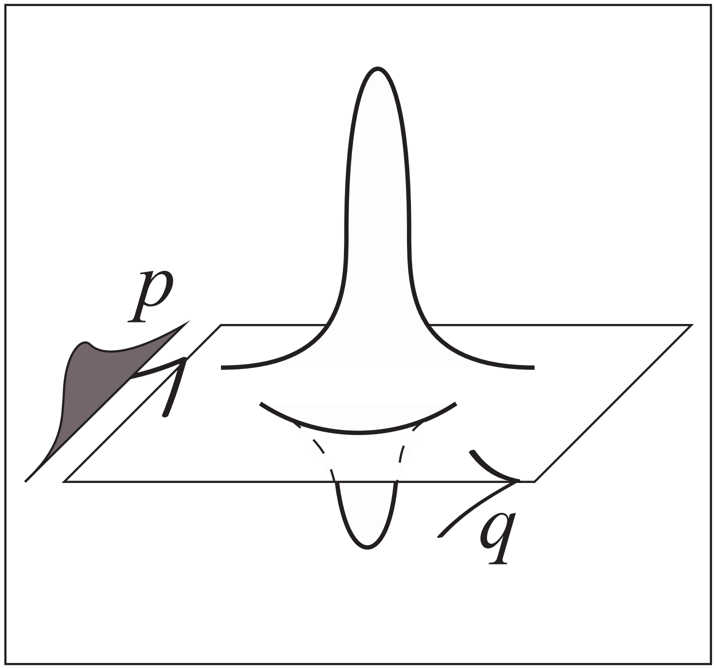
Wigner function

- The Wigner function

$$W_\psi(p,q) = \frac{1}{\pi}\int d\xi\, e^{-2\pi i \xi p}\psi^\dagger(q-\xi/2)\psi(q+\xi/2).$$

is a quasi-probability distribution.

[quantum] mechanics in phase space

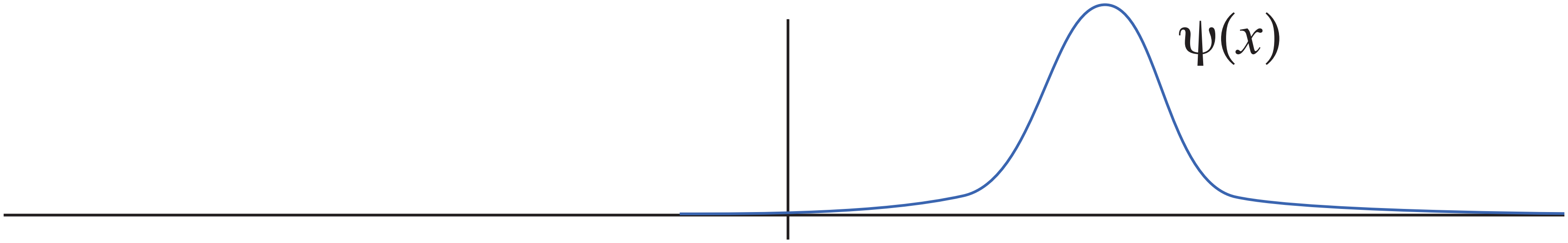


- Wigner function can go negative
- Marginals must be non-negative

Wigner function negativity is an indicator of quantumess

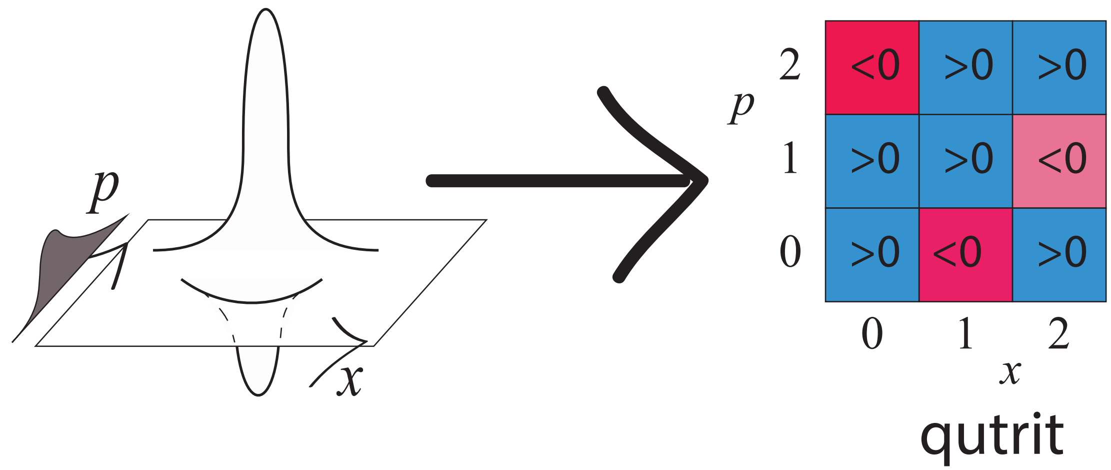
Which states have positive/ negative Wigner function?

Hudson's theorem



Theorem. A pure state ψ has a non-negative Wigner function if and only if and only if ψ is Gaussian, i.e. $\psi(x) \sim e^{2\pi i(x\theta x + ax)}$.

Wigner functions for qudits



Wigner functions can be adapted to finite-dimensional state spaces.

- The Wigner function W is linear in ρ.
- The marginals of W are probability distributions.
- W is informationally complete.

Hudson's theorem for qudits

If the local Hilbert space dimension d is an odd prime, then

Theorem.* [discrete Hudson] A pure state $\psi \in \mathcal{H}_d^{\otimes n}$ has a positive Wigner function if and only if it is a *stabilizer state*.

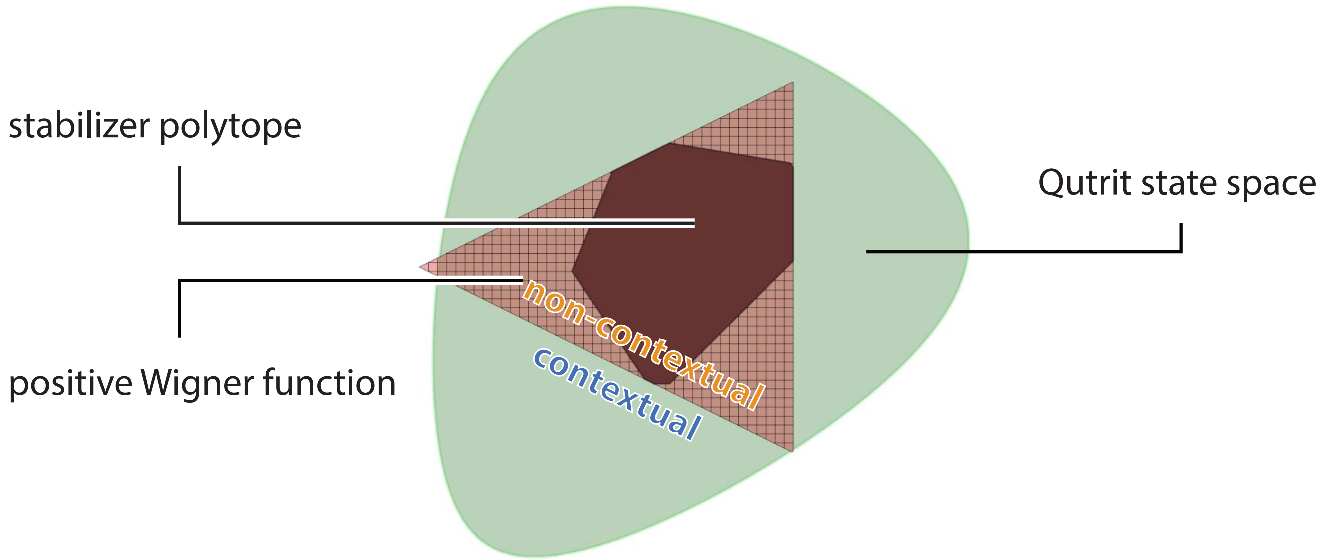
Thus, pure stabilizer states are classical because

1. They have non-negative Wigner function.
2. They can be efficiently simulated (Gottesman-Knill).

*: D. Gross, PhD thesis, 2005.

Quantum computation by state injection

The case of odd prime local Hilbert space dimension



- Clifford operations cannot introduce negativity
- Set of positive states = set of non-contextual states
- Clifford operations cannot introduce contextuality

Contextuality, Wigner negativity: necessary resources for QC.

M. Howard *et al.*, Nature 510, 351 (2014)

Negativity and contextuality in quantum computation

Local Hilbert space dimension $d = 2$

The trouble with $d = 2$

- The standard Wigner function

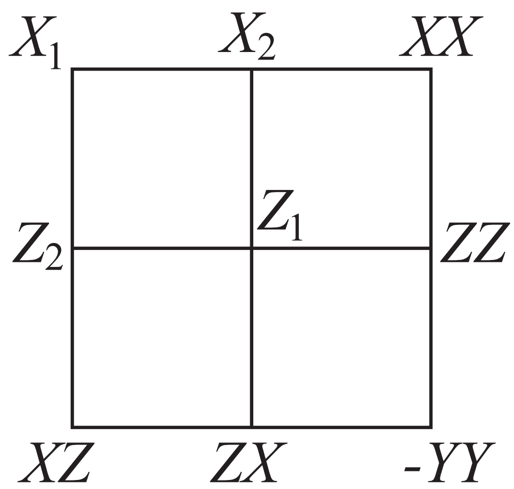
$$W_\psi(p,q) = \frac{1}{\pi}\int d\xi\, e^{-2\pi i \xi p}\psi^\dagger(q - \xi/2)\psi(q + \xi/2).$$

  requires the existence of an inverse of 2 in $\mathbb{F}_d$.

- Does not work in $d = 2$

$\Rightarrow$ Require a different definition of the Wigner function.

The trouble with $d = 2$



- Mermin's square: for multiple qubits, have *state-independent contextuality* w.r.t. Pauli measurements.

$\Rightarrow$ Not all contextuality present can be attributed to states.

- Worse: *Mermin's square yields contextuality witness that classifies all 2-qubit quantum states as contextual.*

Switching to rebits

We make two changes:

1. At all stages, the density matrix ρ of the processed quantum state is *real* w.r.t. the computational basis,

$$\rho = \left(\rho_{ij}\right), \quad \rho_{ij} = \rho_{ji} \in \mathbb{R}.$$

2. The Clifford gates are replaced by the *CSS-ness preserving Clifford gates* as the restricted gate set.

Note that this does not immediately alleviate the problems:

- The local Hilbert space dimension is still $d = 2$.
- The (rotated) Mermin square embeds into real quantum mechanics.

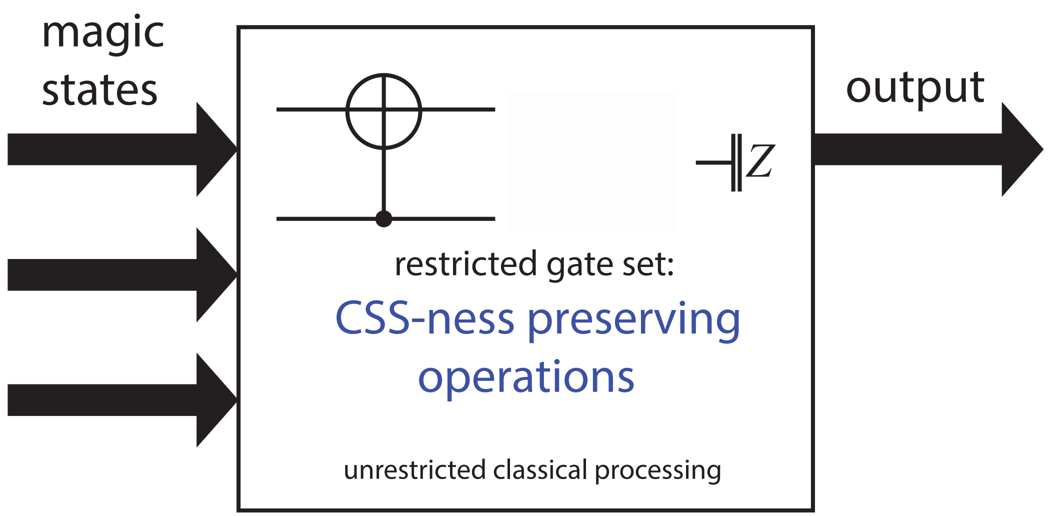
Tasks

1. Devise universal scheme of quantum computation by state injection on rebits

2. Construct matching Wigner function

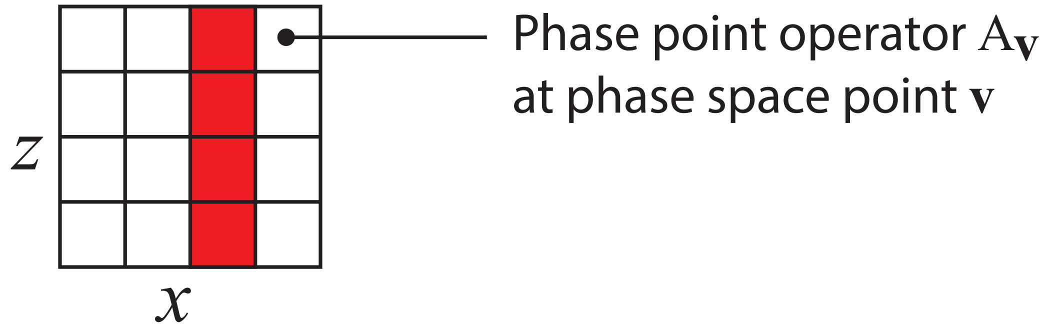
3. Find matching notion of state-dependent contextuality & establish it as necessary resource

1. The computational scheme



- Non-universal gate set:
  - *CSS-ness preserving Clifford gates,*
  - *Measurement of Pauli operators $X(\mathbf{a}_X)$, $Z(\mathbf{a}_Z)$,*
  - *Preparation of CSS-states.*
- Universality reached through injection of *magic states*.
- Encode n qubits in $n+1$ rebits.

2. Rebit Wigner function W_ρ



W is built from Pauli/ translation operators $T_\mathbf{a} = Z(\mathbf{a}_Z)X(\mathbf{a}_X)$:

$$W_\rho(\mathbf{v}) = \frac{1}{2^n}\mathrm{Tr} A_\mathbf{v}\rho, \quad \forall \mathbf{v} \in \mathbb{Z}_2{}^n \times \mathbb{Z}_2{}^n, \tag{1}$$

where

$$A_0 = \frac{1}{2^n} \sum_{\mathbf{v}|\ \mathbf{v}_Z \cdot \mathbf{v}_X = 0} 1\, T_\mathbf{v}. \tag{2}$$

and

$$A_\mathbf{v} = T_\mathbf{v} A_0 T_\mathbf{v}^\dagger, \tag{3}$$

2. Properties of the rebit Wigner function W_ρ

1. W_ρ is informationally complete for real ρ,

$$\rho = \sum_{\mathbf{u}} W_\rho(\mathbf{u}) A_{\mathbf{u}}. \tag{4}$$

2. The trace inner product is given as

$$\mathrm{Tr}\rho\sigma = 2^n \sum_{\mathbf{u}\in\mathbb{Z}_2^{2n}} W_\rho(\mathbf{u}) W_\sigma(\mathbf{u}). \tag{5}$$

3. For all real density matrices ρ, σ,

$$W_{\rho\otimes\sigma} = W_\rho \cdot W_\sigma. \tag{6}$$

2. Properties of the rebit Wigner function W_ρ

Theorem [$d = 2$ Hudson] A pure n-rebit state has a non-negative Wigner function if and only if it is a *CSS* stabilizer state.

$\Rightarrow$ *This is why CSS-ness preserving Clifford gates are chosen as restricted gate set!*

3. Non-negativity implies non-contextuality

Lemma. $W_\rho \geq 0 \longrightarrow$ Pauli measurements on ρ are described by a non-contextual HVM.

Proof sketch: A positive Wigner function *is* a non-contextual HVM.

Consider a POVM with elements E_a. The probability of outcome a is

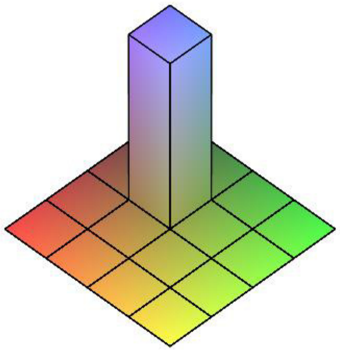
$$p_a := \mathrm{Tr} E_a \rho = 2^n \sum_{\mathbf{u} \in \mathbb{Z}_2^{2n}} W_{E_a}(\mathbf{u}) W_\rho(\mathbf{u}).$$

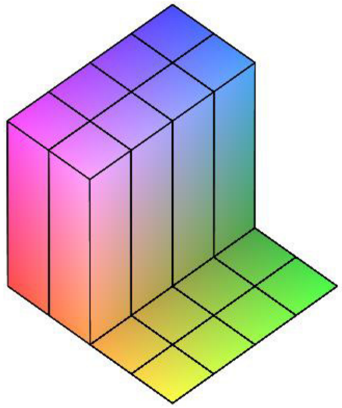
For the allowed measurements, all $W_{E_a} \geq 0$. Therefore may identify

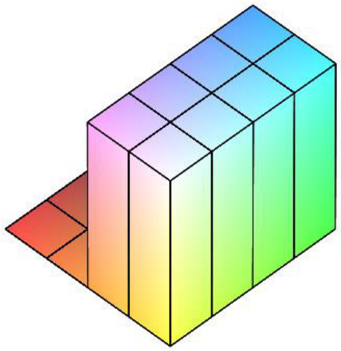
$$\begin{array}{rcl} \{\mathbf{u} \in \mathbb{Z}_2^{2n}\} & : & \text{set of states} \\ W_\rho(\mathbf{u}) & : & \text{probability of state } \mathbf{u} \\ 2^n W_{E_a} & : & \text{conditional probability of outcome } a \text{ given } \mathbf{u}. \end{array}$$

Have a non-contextual HVM.

... meanwhile under the rug



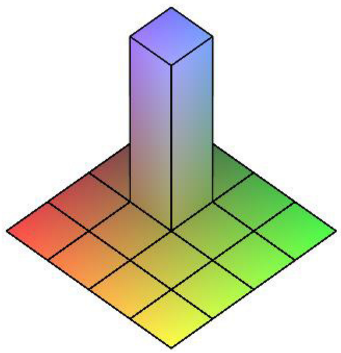




Wigner function of classical state **u**	Wigner function of effect $E_+ = \frac{I+X_1}{2}$	Wigner function of effect $E_- = \frac{I-X_1}{2}$
$W_{\mathbf{u}}(\mathbf{v}) = \delta_{\mathbf{u},\mathbf{v}}$	$2^n W_{E_+} = \delta_{x_1,0}$	$2^n W_{E_-} = \delta_{x_1,1}$
probability for **u**	conditional probability for outcome=+1	conditional probability for outcome=-1

$\Rightarrow$ For every **u**, every real Pauli observable has a value ± 1.

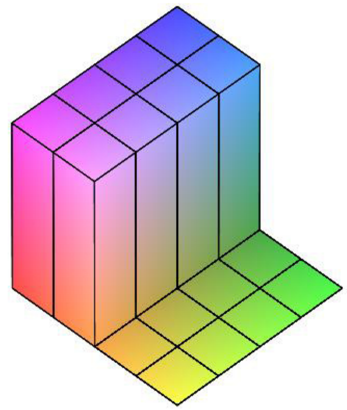
... meanwhile under the rug



Wigner function of
classical state **u**

$W_{\mathbf{u}}(\mathbf{v}) = \delta_{\mathbf{u},\mathbf{v}}$

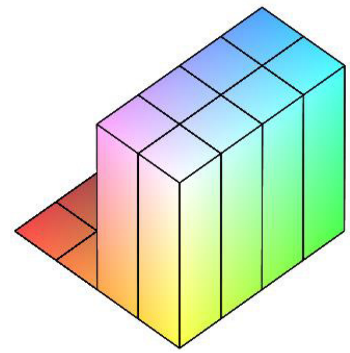
probability for **u**



Wigner function of
effect $E_+ = \frac{I+X_1}{2}$

$2^n W_{E_+} = \delta_{x_1,0}$

conditional probability
for outcome=+1



Wigner function of
effect $E_- = \frac{I-X_1}{2}$

$2^n W_{E_-} = \delta_{x_1,1}$

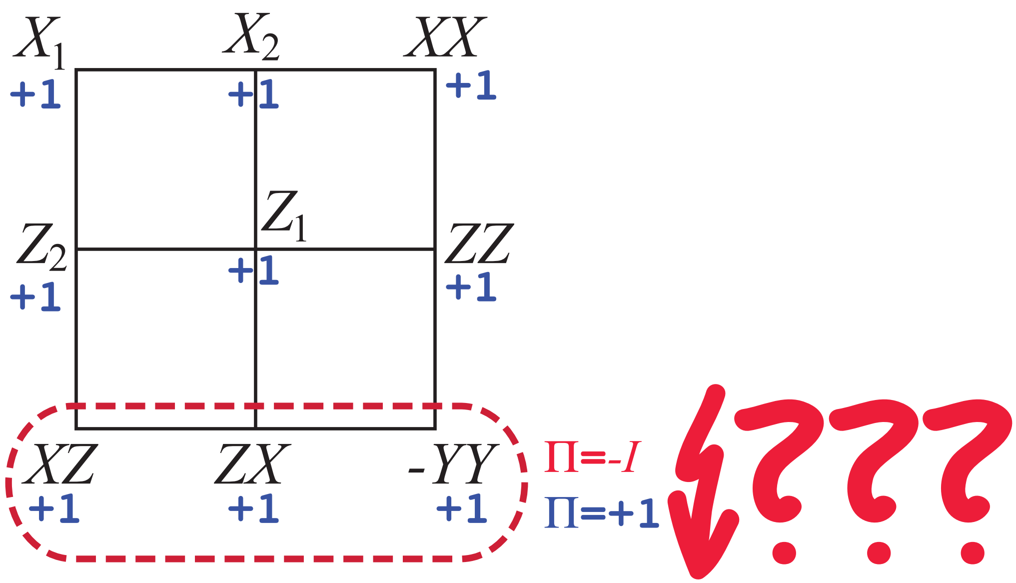
conditional probability
for outcome=-1

$\Rightarrow$ For every **u**, every real Pauli observable has a value ± 1.

How does that fit with Mermin's square?

3. No contradiction with Mermin's square

Value assignment for $\mathbf{u} = \mathbf{0}$: Value $+1$ for all real $T_{\mathbf{v}}$.



- However, value assignments *need not* be consistent in the context $(XZ, ZX, -YY)$.

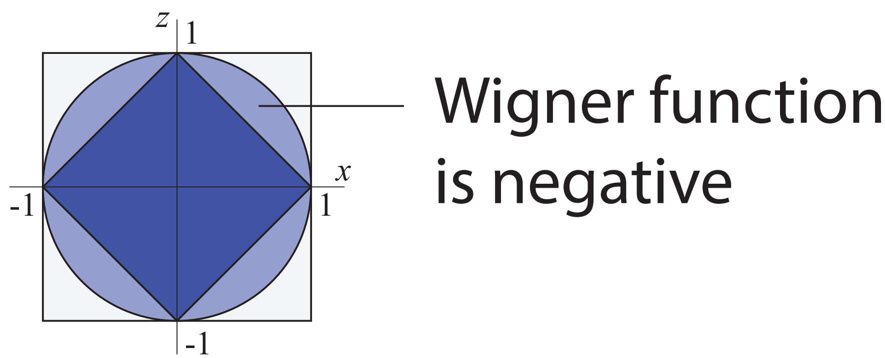
- The observables ZX and XZ cannot be simultaneously measured in the computational scheme.

- Only all-X or all-Z Pauli operators can be physically measured.

3. Negativity does not imply contextuality

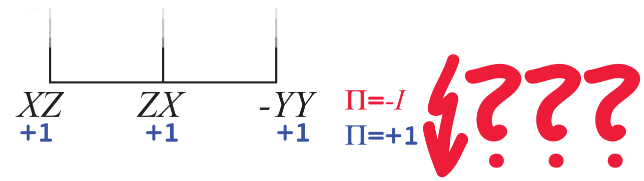
Consider the single-rebit state

$$\rho = \frac{I + xX + zZ}{2}$$



- All states ρ are non-contextual. An explicit hidden-variable model can be constructed for them.

3. Contextuality as resource



The expectation $\langle I + XZ + ZX - YY \rangle$ is a *contextuality witness*. Namely, if

$$\mathcal{W}_\rho = \langle I + XZ + ZX - YY \rangle_\rho < 0,$$

then ρ is contextual.

Proof: Consider HVM state $\mathbf{u}$ with value assignment λ. Then

$$\begin{aligned}
\lambda(XZ) &= \lambda(X_1)\lambda(Z_2),\\
\lambda(ZX) &= \lambda(Z_1)\lambda(X_2),\\
\lambda(-YY) &= \lambda(XX)\lambda(ZZ) = \lambda(X_1)\lambda(X_2)\lambda(Z_1)\lambda(Z_2)
\end{aligned}$$

Therefore, for the witness $\mathcal{W}$ applied to an HVM state $\mathbf{u}$,

$$\begin{aligned}
\mathcal{W}_{\mathbf{u}} &= 1 + \lambda(X_1)\lambda(Z_2) + \lambda(Z_1)\lambda(X_2) + \lambda(X_1)\lambda(X_2)\lambda(Z_1)\lambda(Z_2)\\
&= (1 + \lambda(X_1)\lambda(Z_2))(1 + \lambda(Z_1)\lambda(X_2))\\
&\geq 0.
\end{aligned}$$

□

3. Contextuality as resource

- The contextuality witness $\mathcal{W}_\rho = \langle I + XZ + ZX - YY \rangle_\rho$ can indeed take negative values.

  Consider $\rho = |G\rangle\langle G|$, with $|G\rangle$ a 2-qubit graph state, such that $XZ\,|G\rangle = ZX\,|G\rangle = -|G\rangle$. $\boxed{\text{Thus, } \mathcal{W}_{|G\rangle\langle G|} = -2}$.

- A very large class of contextuality witnesses can be defined, such that this class is mapped onto itself under all CSS-ness preserving Clifford unitaries.

$\Rightarrow$ Contextuality is only maintained or destroyed (measurement), *but never created* in CSS-ness preserving operations.

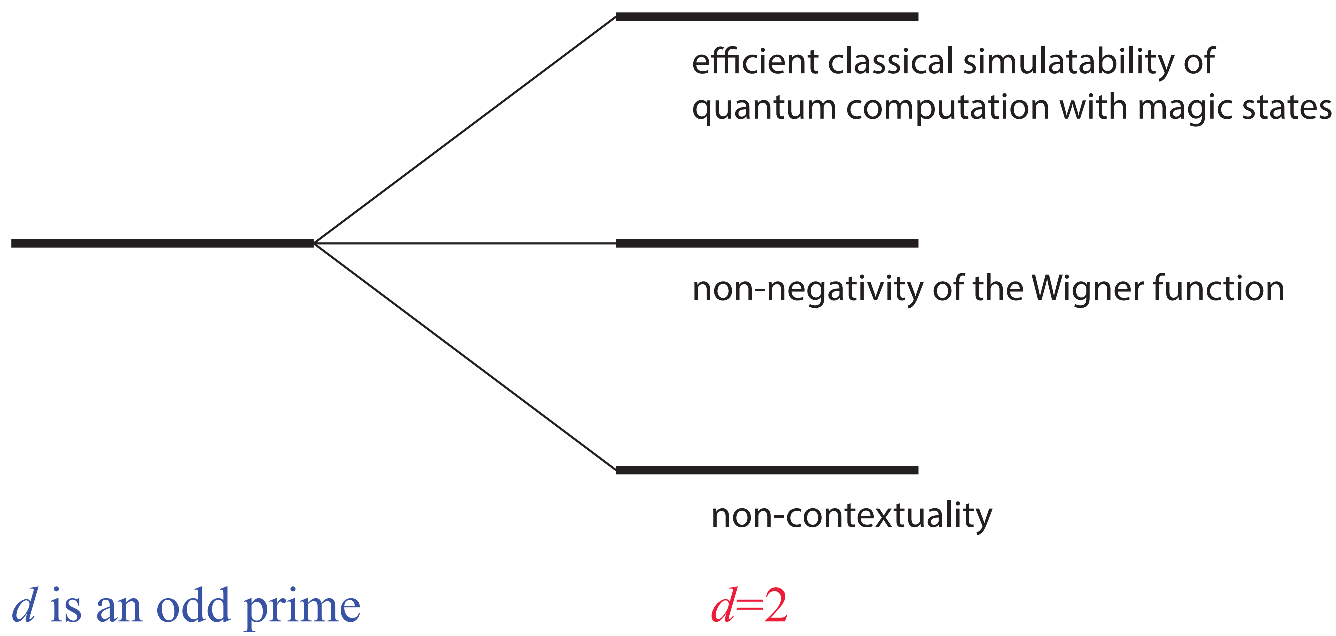
$\Rightarrow$ All contextuality must come from the initial magic states [=Resource].

Results

- Contextuality and negativity are necessary resources in quantum computation with magic states on rebits.
- State-independent contextuality, as it appears for example in Mermin's square and star, is *not* an obstacle.

arXiv:1409.5170

Open questions



- *Three notions of classicality collapse into one for d odd, but not for $d = 2$. Why is that?*

1. Computational scheme

- Use encoding of n qubits in $n+1$ rebits*:

$$|\Psi\rangle \longrightarrow \mathcal{R}(|\Psi\rangle) \otimes |R\rangle_{n+1} + \mathcal{I}(|\Psi\rangle) \otimes |I\rangle_{n+1}.$$

- Restricted gate set: CSS-ness preserving operations

$$CNOTs,\ H_{\text{all}},\ \text{Pauli flips, measurements of } Z_i, X_i.$$

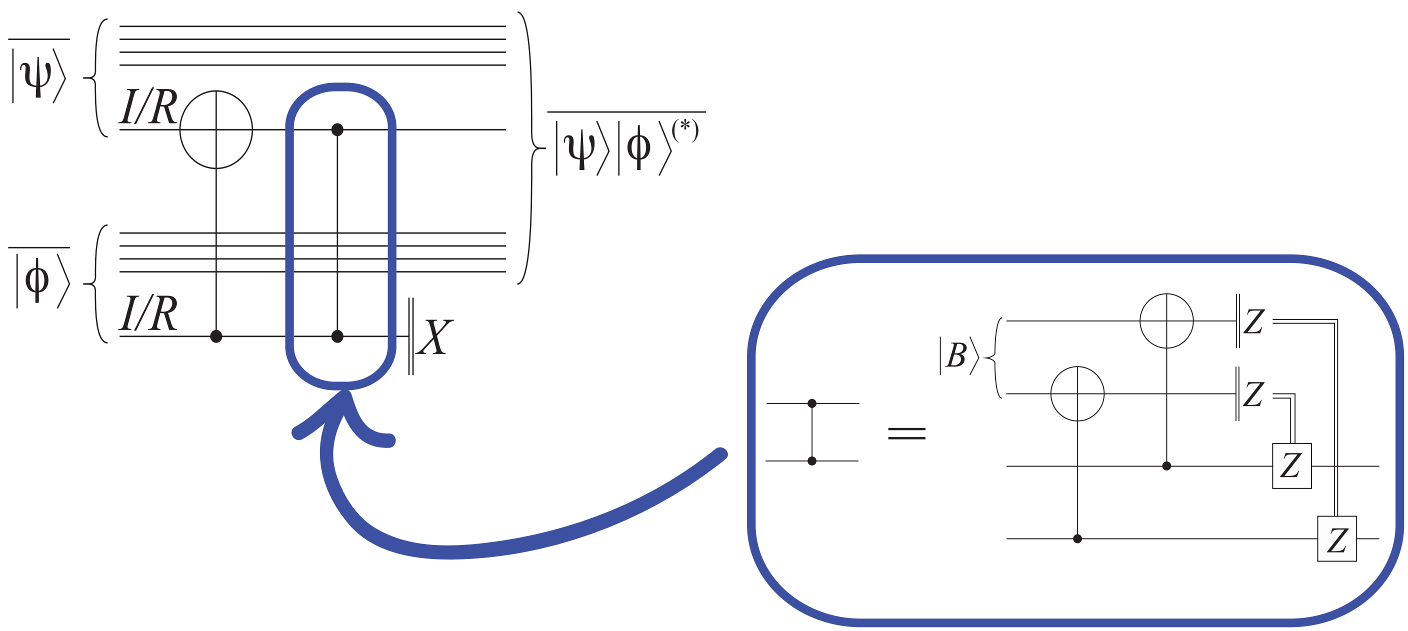
- Use magic states

$$\begin{aligned} |A\rangle &= \frac{|0\rangle|0\rangle}{\sqrt{2}} + |1\rangle\frac{|0\rangle+|1\rangle}{2} \\ |B\rangle &= \frac{|0\rangle|+\rangle+|1\rangle|-\rangle}{\sqrt{2}} \end{aligned}.$$

[*] T. Rudolph and L. Grover, *Encoded universality using rebits*, quant-ph/02

1. Computational scheme

- Devise circuits for the various encoded gates.



Example: circuit for code merging

Purpose: Merge separately encoded ancillas into one code block.